

# NOETHERIAN LOOP SPACES

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**ABSTRACT.** The class of loop spaces whose mod  $p$  cohomology is Noetherian is much larger than the class of  $p$ -compact groups (for which the mod  $p$  cohomology is required to be finite). It contains Eilenberg-Mac Lane spaces such as  $\mathbb{C}P^\infty$  and 3-connected covers of compact Lie groups. We study the cohomology of the classifying space  $BX$  of such an object and prove it is as small as expected, that is, comparable to that of  $B\mathbb{C}P^\infty$ . We also show that  $BX$  differs basically from the classifying space of a  $p$ -compact group in a single homotopy group. This applies in particular to 4-connected covers of classifying spaces of Lie groups and sheds new light on how the cohomology of such an object looks like.

## INTRODUCTION

From the point of view of homotopy theory compact Lie groups are finite loop spaces, i.e. triples  $(X, BX, e)$  where  $X$  is a finite complex, and  $e : X \rightarrow \Omega BX$  is a homotopy equivalence. Most of their geometric features are captured  $p$ -locally in homotopy theory, where  $p$  is any prime, as shown by Dwyer and Wilkerson in [18]. They introduced the notion of  $p$ -compact group, replacing the finiteness condition by a cohomological one, namely that  $H^*(X; \mathbb{F}_p)$  must be finite, and requiring the additional property that  $BX$  be local with respect to mod  $p$  homology, or equivalently  $p$ -complete, [5]. It is the “classifying space”  $BX$  which carries all of the information about the loop space. Amazingly enough, apart from compact Lie groups, there are only a few families of exotic  $p$ -compact groups and they have been recently completely classified by Andersen, Grodal, Møller, and Viruel, see [3] for the odd prime case and [2], [32], [33] for the prime 2 (the only exotic 2-compact group is basically the space  $DI(4)$  constructed by Dwyer and Wilkerson, [17]).

If one aims at an understanding not only of finite loop spaces, but larger ones, the next natural step to take is to relax the cohomological finiteness condition. We define thus a  *$p$ -Noetherian group* to be a loop space  $(X, BX, e)$  where  $BX$  is  $p$ -complete and  $H^*(X; \mathbb{F}_p)$  is a finitely generated (Noetherian)  $\mathbb{F}_p$ -algebra. Relying on Bousfield localization techniques, [4], and Miller’s solution to the Sullivan conjecture, [30], more precisely on Lannes’  $T$ -functor technology, [28], we describe the structure of  $p$ -Noetherian groups and their relation to  $p$ -compact groups. We compute qualitatively

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the cohomology of the classifying space  $BX$  and obtain new general results when  $X$  is the 3-connected cover of a compact Lie group, see Corollary 5.9.

Let us be more precise. In the case of  $p$ -compact groups, i.e. when the mod  $p$  cohomology of the loop space  $H^*(X; \mathbb{F}_p)$  is finite, Dwyer and Wilkerson's main theorem in [18] shows that there are severe restrictions on the cohomology of the classifying space:  $H^*(BX; \mathbb{F}_p)$  is always a finitely generated (Noetherian)  $\mathbb{F}_p$ -algebra. Likewise the cohomology of the classifying space of a  $p$ -Noetherian group cannot be arbitrarily large.

**Theorem 5.1** *Let  $(X, BX, e)$  be a  $p$ -Noetherian group. Then  $H^*(BX; \mathbb{F}_p)$  is finitely generated as an algebra over the Steenrod algebra.*

This information is not optimal as it does not permit to decide for example when an Eilenberg-Mac Lane space of type  $K(\mathbb{Z}/p, m)$  is a  $p$ -Noetherian group. The mod  $p$  cohomology of any of them is finitely generated as an algebra over  $\mathcal{A}_p$ , but the only classifying spaces of a  $p$ -Noetherian group are  $K(\mathbb{Z}/p, 1)$  and  $K(\mathbb{Z}/p, 2)$ . Schwartz's Krull filtration of the category  $\mathcal{U}$  of unstable modules is an established and convenient tool to measure how large an unstable algebra is, [35]. The Krull filtration  $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \dots$  is defined inductively, starting with the full subcategory  $\mathcal{U}_0$  of  $\mathcal{U}$  of locally finite unstable modules (the span of every element under the action of the Steenrod algebra is finite). In fact the cohomology  $H^*(X; \mathbb{F}_p)$  is locally finite if and only if the space  $X$  is  $B\mathbb{Z}/p$ -local, i.e. the evaluation map  $\text{map}(B\mathbb{Z}/p, X) \rightarrow X$  is a weak equivalence ([29, Théorème 0.14]).

There are many  $B\mathbb{Z}/p$ -local spaces, but there are none for which the cohomology lies in higher stages of the Krull filtration. This is the statement of Kuhn's non-realizability conjecture [27], which has been settled by Schwartz in [36] and [37], and proved in its full generality by Dehon and Gaudens in [11]. Thus the cohomology of a space lies in  $\mathcal{U}_0$  or it does not lie in any  $\mathcal{U}_n$ . The cohomology  $H^*(K(\mathbb{Z}/p, m); \mathbb{F}_p)$  for example does not lie in any  $\mathcal{U}_n$ , but the quotient module of indecomposable elements  $QH^*(K(\mathbb{Z}/p, m); \mathbb{F}_p)$  lies in  $\mathcal{U}_{m-1}$  for any  $m \geq 1$  ([8, Example 2.2]).

It was observed in [8, Lemma 7.1] that if  $H^*(BX; \mathbb{F}_p)$  is finitely generated as an algebra over  $\mathcal{A}_p$ , then  $QH^*(BX; \mathbb{F}_p)$  must be finitely generated as a module over  $\mathcal{A}_p$ , and hence lies in  $\mathcal{U}_k$  for some  $k$ . We remark that the condition that  $H^*(X; \mathbb{F}_p)$  be a Noetherian  $\mathbb{F}_p$ -algebra is equivalent to saying that  $H^*(X; \mathbb{F}_p)$  is finitely generated as an algebra over  $\mathcal{A}_p$  and that the unstable module  $QH^*(X; \mathbb{F}_p)$  of indecomposable elements lies in  $\mathcal{U}_0$ . Our second result shows that the cohomology of the classifying space of a  $p$ -Noetherian group is as small as expected in terms of the Krull filtration.

**Theorem 3.3** *Let  $(X, BX, e)$  be a  $p$ -Noetherian group. Then  $QH^*(BX; \mathbb{F}_p)$  belongs to  $\mathcal{U}_1$ .*

The arguments to prove these results are the following. We start in Section 1 with the study of the structure of  $p$ -Noetherian groups. The most basic examples of  $p$ -Noetherian groups  $X$  are  $p$ -compact groups and Eilenberg-Mac Lane spaces  $K(\mathbb{Z}/p^r, 1)$  and  $K(\mathbb{Z}_p^\wedge, 2)$ . In the spirit of our

deconstruction results for  $H$ -spaces, [8], we show that these are the basic building blocks for all  $p$ -Noetherian groups.

**Theorem 1.9** *Let  $(X, BX, e)$  be any  $p$ -Noetherian group. There exists then a fibration*

$$K(P, 2)_p^\wedge \longrightarrow BX \longrightarrow BY,$$

where  $P$  is a  $p$ -group which is a finite direct sum of copies of cyclic groups and Prüfer groups and  $Y$  is a  $p$ -compact group.

We stress that this fibration is functorial in  $BX$  as the base space is obtained by Bousfield localization. The understanding of the cohomology of  $BX$  goes through the analysis of the Serre spectral sequence of this fibration. Note that, by Dwyer and Wilkerson's main theorem in [18],  $H^*(BY; \mathbb{F}_p)$  is finitely generated as an algebra. Also, the mod  $p$  cohomology of  $K(P, 2)_p^\wedge$  is finitely generated as an algebra over the Steenrod algebra by [38] and [7]. The spectral sequence is not nearly as nice as what happens for  $H$ -spaces though, compare with [9], and we must first tackle Theorem 3.3.

This we do in Sections 2 – 3 by giving first a geometric interpretation to  $\bar{T}QH^*(BX; \mathbb{F}_p)$ , where  $\bar{T}$  is Lannes' reduced  $T$ -functor. Recall that for “nice” spaces (such as  $BX$ ) the unreduced  $T$  functor  $TH^*(BX; \mathbb{F}_p)$  computes the cohomology of the mapping space  $\text{map}(B\mathbb{Z}/p, BX)$  and we rely on Schwartz's characterization [35, Theorem 6.2.4] of the Krull filtration in terms of  $\bar{T}$ . In order to perform our reduced  $T$  functor calculation, we prove that the component  $\text{map}(B\mathbb{Z}/p, BX)_c$  of the constant map splits as a product  $BX \times \text{map}_*(B\mathbb{Z}/p, BX)_c$ . Using the properties of the  $T$  functor, this splitting yields a geometric interpretation of the reduced  $T$  functor in terms of the pointed mapping space, more precisely  $\bar{T}QH^*(X; \mathbb{F}_p) \cong H^*(\text{map}_*(B\mathbb{Z}/p, BX)_c; \mathbb{F}_p)$ .

We finally come back to Theorem 5.1, which we prove in two steps. First we investigate in Section 4 the Serre Spectral sequence for fibrations over spaces with finite cohomology and fibre a finite product of Eilenberg Mac–Lane spaces. We show that in this situation the cohomology of the total space is finitely generated as an algebra over  $\mathcal{A}_p$ . Secondly, we reduce the situation in which the base space of the fibration is a  $p$ -compact toral group.

Let us conclude the introduction with two remarks. The results in this article show that for a compact simply-connected Lie group  $G$ , the module  $QH^*((BG)\langle 4 \rangle; \mathbb{F}_p)$  is finitely generated over  $\mathcal{A}_p$  and belongs to  $\mathcal{U}_1$ . This puts into context the calculations made by Harada and Kono, [24], and sheds new light on how the cohomology will look like even in the cases where an explicit description has not been obtained, compare with Example 5.10.

Let us finally mention Corollary 4.6 in which we obtain, as a byproduct of our theory, a description of the cohomology of the  $n$ -connected cover of an  $(n - 1)$ -connected finite complex. This was only known for  $H$ -spaces before, [9].

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## 1. THE STRUCTURE OF $p$ -NOETHERIAN GROUPS

This first section is devoted to the description of the classifying space of  $p$ -Noetherian groups and the relation to  $p$ -compact groups. Let us start with the definition and the basic examples. The statements about the action of the Steenrod algebra and the Krull filtration will be explained and developed in Section 3.

**Definition 1.1.** A  $p$ -Noetherian group is a triple  $(X, BX, e)$  where  $H^*(X; \mathbb{F}_p)$  is a Noetherian algebra (it is finitely generated as an algebra),  $BX$  is a  $p$ -complete space, and  $e : X \rightarrow \Omega BX$  is a weak equivalence.

We will often use the word  $p$ -Noetherian group for the loop space  $X$  and refer to  $BX$  as the classifying space. Hence, we will say that a  $p$ -Noetherian group is  $n$ -connected if so is the loop space  $X$ , or equivalently if the classifying space is  $(n + 1)$ -connected. Note that if the *integral* cohomology of a space is Noetherian, as an algebra, then so is the mod  $p$  cohomology (just like finite loop spaces have finite mod  $p$  cohomology).

**Remark 1.2.** Since  $\pi_1(BX) \cong \pi_0(X)$  and  $H^*(X; \mathbb{F}_p)$  is of finite type, it follows that  $\pi_1(BX)$  is finite and, therefore  $BX$  is a  $p$ -good space by [5, VII, Proposition 5.1]. In fact,  $\pi_1(BX)$  is a  $p$ -group by [5, VII, Proposition 4.3].

**Example 1.3.** In [18], Dwyer and Wilkerson introduced the notion of a  $p$ -compact group. A  $p$ -compact group is a loop space  $(X, BX, e)$  such that  $BX$  is  $p$ -complete and  $H^*(X; \mathbb{F}_p)$  is a finite  $\mathbb{F}_p$ -vector space. It is clear from the definition that  $p$ -compact groups are  $p$ -Noetherian groups.

The most basic example of  $p$ -compact group is given by the  $p$ -completed circle and its classifying space  $K(\mathbb{Z}_p^\wedge, 2)$ . Our definition of  $p$ -Noetherian group allows to include not only all  $p$ -compact groups but also the following Eilenberg-Mac Lane spaces.

**Example 1.4.** Let  $X = K(\mathbb{Z}_p^\wedge, 2)$ ,  $BX = K(\mathbb{Z}_p^\wedge, 3)$ , and  $e$  the obvious homotopy equivalence between  $\Omega K(\mathbb{Z}_p^\wedge, 3)$  and  $K(\mathbb{Z}_p^\wedge, 2)$ . This is a  $p$ -Noetherian group since  $H^*(K(\mathbb{Z}_p^\wedge, 2); \mathbb{F}_p) \cong \mathbb{F}_p[u]$  is finitely generated as an algebra. Let us point out here that  $H^*(K(\mathbb{Z}_p^\wedge, 3); \mathbb{F}_p)$  is finitely generated as an algebra over  $\mathcal{A}_p$  and that the module of indecomposable elements  $QH^*(K(\mathbb{Z}_p^\wedge, 3); \mathbb{F}_p)$  lives in  $\mathcal{U}_1$ . For example  $QH^*(K(\mathbb{Z}_2^\wedge, 3); \mathbb{F}_2) \cong \Sigma F(1)$ , where  $F(1)$  is the free unstable module on one generator in degree 1.

In fact, this is basically the only 1-connected  $p$ -Noetherian group  $X$  such that  $\Omega X$  is  $\mathbb{F}_p$ -finite.

**Proposition 1.5.** *Let  $(X, BX, e)$  be a  $p$ -Noetherian group such that  $H^*(\Omega X; \mathbb{F}_p)$  is finite. Then  $BX$  is 2-connected if and only if it is a product of a finite number of copies of  $K(\mathbb{Z}_p^\wedge, 3)$ .*

*Proof.* The loop space  $\Omega X \simeq \Omega^2 BX$  is a connected homotopy commutative mod  $p$  finite  $H$ -space. Thus, by the mod  $p$  version of Hubbuck's Torus Theorem, [25] and [1], we see that  $\Omega X$  is equivalent to a  $p$ -completed torus. ■

**Example 1.6.** Let us consider the compact Lie group  $S^3$  and its 3-connected cover  $S^3\langle 3 \rangle$ . Identifying  $B(S^3\langle 3 \rangle)$  with  $(BS^3)\langle 4 \rangle$ , we have a fibration  $K(\mathbb{Z}, 3) \rightarrow (BS^3)\langle 4 \rangle \rightarrow BS^3$ . The triple  $(S^3\langle 3 \rangle_p^\wedge, B(S^3\langle 4 \rangle_p^\wedge), e)$  is then a  $p$ -Noetherian group since  $H^*(S^3\langle 3 \rangle; \mathbb{F}_p) \cong \mathbb{F}_p[x] \otimes E(y)$  where  $x$  has degree  $2p$ ,  $y$  has degree  $2p+1$ , and a Bockstein connects  $x$  and  $y$ ,  $\beta(x) = y$ . This  $p$ -Noetherian group is an extension of the  $p$ -compact group  $(S^3)_p^\wedge$  and the Eilenberg-Mac Lane space from Example 1.4.

It is not difficult to compute the mod  $p$  cohomology of  $(BS^3)\langle 4 \rangle$  using the Serre spectral sequence. For example,  $H^*(B(S^3\langle 3 \rangle); \mathbb{F}_2) \cong \mathbb{F}_2[z, Sq^1 z, Sq^4 z, Sq^{8,4} z, \dots]$ , where  $z$  has degree 5. This is a subalgebra of the mod 2 cohomology of  $K(\mathbb{Z}, 3)$  and  $z$  corresponds to  $Sq^2 \iota_3$ , where  $\iota_3$  is the fundamental class. Again, we see that the module of indecomposable elements belongs to  $\mathcal{U}_1$ , as it differs from  $\Sigma F(1)$  by only a few classes in low degrees.

This last example fits into a more general picture. One can consider 4-connected covers of classifying spaces of compact Lie groups.

**Example 1.7.** Let  $G$  be a simply connected compact Lie group and consider the 4-connected cover of its classifying space,  $(BG)\langle 4 \rangle$ . Since the mod  $p$  cohomology  $G\langle 3 \rangle$  is Noetherian, this provides an infinite number of examples of  $p$ -Noetherian groups.

Harada and Kono studied in [24] and [23] the fibration  $K(\mathbb{Z}, 3) \rightarrow (BG)\langle 4 \rangle \rightarrow BG$ . They were able to compute explicitly, as an algebra, the cohomology of the total space at odd primes and a few cases at the prime 2. In all of these computations the result is the tensor product of a quotient of  $H^*(BG; \mathbb{F}_p)$  with a certain subalgebra of  $H^*(K(\mathbb{Z}, 3); \mathbb{F}_p)$ , which turns out to be always finitely generated as an algebra over the Steenrod algebra.

In fact,  $p$ -Noetherian groups are closed under fibrations in the following sense (and this explains why  $G\langle 3 \rangle$  defines a  $p$ -Noetherian group).

**Proposition 1.8.** *Let  $BX \rightarrow E \rightarrow BZ$  be a fibration of connected spaces where  $BX$  and  $BZ$  are classifying spaces of  $p$ -Noetherian groups. Then  $E$  is also the classifying space of a  $p$ -Noetherian group.*

*Proof.* Since  $\pi_1(BZ)$  is a finite  $p$ -group and both  $BX$  and  $BZ$  are  $p$ -complete and  $p$ -good (see Remark 1.2), the fibre lemma [5, II.5.1] shows that  $E$  is also  $p$ -complete. It remains to show that  $H^*(\Omega E; \mathbb{F}_p)$  is a finitely generated algebra.

Looping the fibration, we obtain an  $H$ -fibration  $X \rightarrow \Omega E \rightarrow Z$  where both  $X$  and  $Z$  have finitely generated mod  $p$  cohomology. By [9, Theorem 5.1], the cohomology of  $\Omega E$  is finitely generated as an algebra over the Steenrod algebra, in other words  $QH^*(\Omega E; \mathbb{F}_p)$  is finitely generated as an  $\mathcal{A}_p$ -module. This unstable module is thus finite if and only if it is locally finite, which by [16] is equivalent to the evaluation map  $\text{map}(B\mathbb{Z}/p, \Omega E)_c \rightarrow E$  to be an equivalence. This follows from the fact that this is the case for  $X$  and  $Z$ . ■

Let us analyze the structure of an arbitrary connected  $p$ -Noetherian group. The following theorem tells us that it always differs from a  $p$ -compact group in a single  $p$ -completed Eilenberg-MacLane space.

**Theorem 1.9.** *Let  $(X, BX, e)$  be any  $p$ -Noetherian group. There exists then a fibration*

$$K(P, 2)_p^\wedge \rightarrow BX \rightarrow BY,$$

where  $P$  is a finite direct sum of copies of cyclic groups and Prüfer groups and  $Y$  is a  $p$ -compact group.

*Proof.* By assumption the mod  $p$  cohomology of  $X$  is finitely generated as an algebra. In other words, the module of indecomposable elements  $QH^*(X; \mathbb{F}_p)$  is finite. Therefore, by [16, Theorem 3.2] the loop space  $\Omega X$  is  $B\mathbb{Z}/p$ -local, or equivalently the classifying space  $BX$  is  $\Sigma^2 B\mathbb{Z}/p$ -local ([20, Theorem 3.A.1]). The analysis in [4] by Bousfield of the Postnikov like nullification tower shows then that the homotopy fiber of the nullification map  $BX \rightarrow P_{\Sigma B\mathbb{Z}/p} BX$  is a single Eilenberg-Mac Lane space  $K(P, 2)$ , where  $P$  is an abelian  $p$ -torsion group. Moreover, he also shows that the corresponding fibration is principal. In particular, it implies that  $P_{\Sigma B\mathbb{Z}/p} BX$  is a  $p$ -good space, and  $(P_{\Sigma B\mathbb{Z}/p} BX)_p^\wedge$  is  $p$ -complete by the fibre lemma [5, II.5.1].

From the equivalence  $P_{B\mathbb{Z}/p} X \simeq \Omega P_{\Sigma B\mathbb{Z}/p} BX$ , [20, Theorem 3.A.1], we obtain a loop fibration  $K(P, 1) \rightarrow X \rightarrow P_{B\mathbb{Z}/p} X$ . Since  $X$  is a loop space with finitely generated mod  $p$  cohomology, we know from [8, Theorem 7.3], or directly from [10], that  $P$  is a finite direct sum of copies of cyclic groups and Prüfer groups and that  $H^*(P_{B\mathbb{Z}/p} X; \mathbb{F}_p)$  is finite.

Let us consider the loop space  $P_{B\mathbb{Z}/p} X$ . Notice that  $\pi_1 P_{\Sigma B\mathbb{Z}/p} BX \cong \pi_1 BX$ , which must be a finite  $p$ -group by Remark 1.2. By  $p$ -completing we obtain hence a  $p$ -compact group  $BY = (P_{\Sigma B\mathbb{Z}/p} BX)_p^\wedge$ . ■

The fibration we have obtained allows us to give a precise description of the component of the constant in the pointed mapping space  $\text{map}_*(B\mathbb{Z}/p, BX)$ .

**Corollary 1.10.** *Let  $(X, BX, e)$  be a  $p$ -Noetherian group. Then  $\text{map}_*(B\mathbb{Z}/p, BX)_c$  is the classifying space of a finite elementary abelian  $p$ -group.*

*Proof.* Consider the fibration  $K(P, 2)_p^\wedge \rightarrow BX \rightarrow BY$  from Theorem 1.9. Since  $BY$  is a  $p$ -compact group,  $H^*(\Omega BY; \mathbb{F}_p)$  is finite and  $\text{map}_*(B\mathbb{Z}/p, \Omega BY) \simeq *$  by [30]. Therefore the component  $\text{map}_*(B\mathbb{Z}/p, BY)_c$  is contractible and  $\text{map}_*(B\mathbb{Z}/p, BX)_c \simeq \text{map}_*(B\mathbb{Z}/p, K(P, 2)_p^\wedge)_c$ . By [30, Theorem 1.5],  $\text{map}_*(B\mathbb{Z}/p, K(P, 2)_p^\wedge) \simeq \text{map}_*(B\mathbb{Z}/p, K(P, 2))$ , which has trivial homotopy groups in degrees  $\geq 2$ . The component of the constant map is thus the classifying space of a finite elementary abelian  $p$ -group  $V = \text{Hom}(\mathbb{Z}/p, P)$ .  $\blacksquare$

From Theorem 1.9 we deduce that many  $p$ -Noetherian groups are 3-connected covers of  $p$ -compact groups.

**Corollary 1.11.** *Let  $(X, BX, e)$  be a  $p$ -Noetherian group. Then  $X$  is 3-connected if and only if  $BX$  is the 4-connected cover of the classifying space of a  $p$ -compact group.*

*Proof.* One implication is obvious. Let us hence assume that  $X$  is 3-connected and consider the fibration  $K(P, 2)_p^\wedge \rightarrow BX \rightarrow BY$  from Theorem 1.9. We see that  $BY$  is 2-connected, hence 3-connected [6, Theorem 6.10]. This shows that  $P$  must be a divisible abelian  $p$ -group, or equivalently that  $K(P, 2)_p^\wedge \simeq K(\oplus \mathbb{Z}_p^\wedge, 3)$ .  $\blacksquare$

To the Lie group Example 1.7 we can add now new examples of  $p$ -Noetherian groups, namely those given by 3-connected covers of exotic  $p$ -compact groups.

**Example 1.12.** Let  $X$  be a  $p$ -compact group such that  $BX$  is 3-connected and  $\pi_4(BX) \cong \mathbb{Z}_p^\wedge$ . By looking at the classification of  $p$ -compact groups, we observe that there are only two sporadic examples, namely numbers 23 and 30 in the Shephard-Todd list [39], and one infinite family, number  $2b$ , corresponding to the dihedral groups  $D_{2m}$ . The triple  $(X\langle 3 \rangle, (BX)\langle 4 \rangle, e)$  is a  $p$ -Noetherian group by Corollary 1.11.

The two sporadic examples are defined at primes  $p \equiv 1, 4 \pmod{5}$ , and they are non-modular since the only primes which divide the order of their Weyl group are 2, 3 and 5. The family of  $p$ -compact groups corresponding to the dihedral groups  $D_{2m}$  is defined for primes  $p \equiv \pm 1 \pmod{m}$ . Note that  $p = 2$  occurs when  $m = 3$  and corresponds to the exceptional Lie group  $G_2$ .

**Remark 1.13.** From Corollary 1.11 we obtain a classification of 3-connected  $p$ -Noetherian groups. They are given by the 3-connected covers of simply connected  $p$ -compact groups, which are known from the recent classification results, [32], [33], [2], and [3]. A general classification will be more difficult to obtain, even in the 2-connected case, as there are  $p$ -Noetherian groups fibering over a product of  $p$ -compact groups which do not split themselves as a product. Consider indeed the homotopy fiber of the composite map

$$f : BS^3 \times BS^3 \rightarrow K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4) \rightarrow K(\mathbb{Z}, 4),$$

where the first map is the fourth Postnikov section and the second is given by the sum. Let us complete this fiber at the prime 7 for example and call it  $BX$ . Even though  $X$  splits as a product  $(S^3)_7^\wedge \times (S^3)_7^\wedge \langle 3 \rangle$ , the classifying space  $BX$  does not split.

Assume that  $BX$  splits as a product  $(BS^3)_7^\wedge \times (BS^3)_7^\wedge \langle 4 \rangle$ . There exists then an essential map  $g : (BS^3)_7^\wedge \rightarrow (BS^3)_7^\wedge \times (BS^3)_7^\wedge$  such that  $f \circ g \simeq *$  and  $p_1 \circ g$  is an equivalence, where  $p_1$  denotes the projection on the first factor. But  $g$  induces on the fourth homology group a map of degree  $n \neq 0$  on the first copy of  $(BS^3)_7^\wedge$  and of degree  $m$  on the second. The composite  $f \circ g$  will thus have degree  $n + m$  on  $H_4$ . We claim that this cannot be zero. Both  $m$  and  $n$  must be squares in  $\mathbb{Z}_7^\wedge$  as a self-map of  $(BS^3)_7^\wedge$  is induced by a self-map on the maximal torus  $(BS^1)_7^\wedge$ . But the sum of two 7-adic squares is nul if and only both are so. Therefore  $BX$  cannot split as a product. We refer the reader to Dwyer and Mislin's article [15] for a complete study of self-maps of  $BS^3$ .

## 2. SPLITTING FIBRATIONS AND MAPPING SPACES

Our next aim is to obtain conditions under which the total space of the evaluation fibration  $\text{map}_*(B\mathbb{Z}/p, BX)_c \rightarrow \text{map}(B\mathbb{Z}/p, BX)_c \rightarrow BX$  splits as a product. The key observation is that  $\text{map}_*(B\mathbb{Z}/p, BX)_c$  has a single non-trivial homotopy group, see Corollary 1.10.

We thus consider a fibration  $F \rightarrow E \rightarrow B$  of connected spaces and assume that the homotopy fiber  $F$  has finitely many homotopy groups (it is a Postnikov piece, or in other words there exists an integer  $n$  such that the  $n$ -th Postnikov section  $F \rightarrow F[n]$  is a homotopy equivalence). Such a fibration is classified by a map  $B \rightarrow B \text{aut}(F)$ , where  $\text{aut}(F)$  denotes the monoid of self-equivalences of  $F$ . The original fibration can be recovered by pulling-back the universal fibration  $F \rightarrow B \text{aut}_*(F) \rightarrow B \text{aut}(F)$ , where  $\text{aut}_*(F)$  is the monoid of pointed self-equivalences. The existence of the universal fibration was known to Dold, [12], but it was Gottlieb who identified first the total space, [22].

**Proposition 2.1.** *Let  $F \simeq F[n]$  be a connected Postnikov piece and  $X$  be any space. Then*

- (1)  $\text{map}(X, F) \simeq (\text{map}(X, F))[n]$  and  $\pi_n(\text{map}(X, F)) \cong \pi_n(F)$ ,
- (2)  $\text{map}_*(X, F) \simeq (\text{map}_*(X, F))[n - 1]$ .

*Proof.* We proceed by induction on the number of non-trivial homotopy groups of  $F$ . When  $F$  is a  $K(G, 1)$ ,  $\text{map}(X, F) \simeq \text{map}(X[1], F)$  and any component has the homotopy type of the classifying space of a centralizer. Likewise  $\text{map}_*(X, K(G, 1))$  is homotopically discrete, the components being in bijection with homomorphisms  $\pi_1 X \rightarrow G$ .

Suppose now that  $n \geq 2$ , write  $A = \pi_n F$ , and consider the fibration  $K(A, n) \rightarrow F \xrightarrow{p} F[n - 1]$ . Let us fix a map  $k : X \rightarrow F$ . We analyze one component of the mapping spaces at a time. Observe first that both claims are true for the component of the constant map since we have a fibration  $\text{map}(X, K(A, n)) \rightarrow \text{map}(X, F) \rightarrow \text{map}(X, F[n - 1])$  and can conclude by the classical result of Thom, [40], and Federer, [21], that  $\text{map}(X, K(A, n))$  splits as a product of Eilenberg-Mac Lane



spaces  $\coprod K(H^i(X; A), n - i)$ . Møller has proved in [31] a relative version of this lemma which will allow us to understand the other components as well.

Let  $\bar{k}$  denote the composite  $X \rightarrow F \rightarrow F[n - 1]$ . The  $(n - 1)$ -st Postnikov section induces a fibration  $\text{map}(X, F)_k \rightarrow \text{map}(X, F[n - 1])_{\bar{k}}$ , the fiber of which is  $F(X, \emptyset; F, F[n - 1])_{\bar{k}}$ , i.e. the space of all lifts  $f : X \rightarrow F$  such that  $p \circ f = \bar{k}$ . This space of lifts is a product of Eilenberg-Mac Lane spaces by [31, Theorem 3.1] just like in the case of the component of the trivial map. In particular the highest non-trivial homotopy group of  $\text{map}(X, F)_k$  is the  $n$ -th, isomorphic to  $A$ .

In the case of pointed mapping spaces the space of lifts  $F(X, *, F, F[n - 1])_{\bar{k}}$  appears in a similar argument. Its highest non-trivial homotopy group is the  $(n - 1)$ -st one. ■

**Remark 2.2.** The proof of Proposition 2.1 also shows that if  $X$  is  $n$ -connected then  $\text{map}_*(X, F)$  is contractible.

When  $F = K(G, 1)$  has a single homotopy group, it is well-known that  $\text{aut}(F)$  is the semi-direct product  $K(G, 1) \ltimes \text{Aut}(G)$  and  $\text{aut}_*(F) \simeq \text{Aut}(G)$ . All higher homotopy groups of the topological monoid of self-equivalences are trivial. The following corollary generalizes this observation.

**Corollary 2.3.** *Let  $n \geq 1$  and  $F \simeq F[n]$  be a connected Postnikov piece. Then*

- (1)  $B \text{aut}(F) \simeq (B \text{aut}(F))[n + 1]$  and  $\pi_{n+1}(B \text{aut}(F)) = \pi_n(F)$ ,
- (2)  $B \text{aut}_*(F) \simeq (B \text{aut}_*(F))[n]$ .

*Proof.* The components of the monoid  $\text{aut}(F)$  are precisely the components in  $\text{map}(F, F)$  of maps which are homotopy equivalences. We conclude by Proposition 2.1. ■

This statement is very much related to the work of Dwyer, Kan, and Smith, [14], where they provide classifying spaces for towers of fibrations with prescribed fibers. In our case these fibers would be  $K(\pi_1 F, 1), \dots, K(\pi_n F, n)$ .

Let us now come back to our fibration  $F \rightarrow E \rightarrow B$ . We ask when the total space  $E$  splits as a product  $B \times F$ .

**Theorem 2.4.** *Let  $F \rightarrow E \rightarrow B$  be a fibration of connected spaces and assume that  $F \simeq F[n]$ .*

- (1) *If  $B$  is  $n$ -connected, then  $E \simeq B \times F$  if and only if the connecting morphism  $\pi_{n+1}(B) \rightarrow \pi_n(F)$  is trivial.*
- (2) *If the fibration has a section  $s : B \rightarrow E$  and  $B$  is  $(n - 1)$ -connected, then  $E \simeq B \times F$  if and only if the morphism  $\pi_n(B) \rightarrow \pi_n(B \text{aut}_*(F))$  is trivial.*

*Proof.* The fibration is classified by a map  $f : B \rightarrow B \text{aut}(F)$  and  $\pi_i B \text{aut}(F) = 0$  for  $i > n + 1$  by Corollary 2.3. Part (1) when  $B$  is  $(n + 1)$ -connected is thus a direct consequence of the first part

of this corollary. Now assume that  $B$  is  $n$ -connected, consider the pullback diagram

$$\begin{array}{ccccc}
 F & \xlongequal{\quad} & F & & \\
 \downarrow & & \downarrow & & \\
 E' & \longrightarrow & E & \longrightarrow & K(\pi_{n+1}(B), n+1) \\
 \downarrow & & \downarrow & & \parallel \\
 B\langle n+1 \rangle & \xrightarrow{i} & B & \longrightarrow & K(\pi_{n+1}(B), n+1)
 \end{array}$$

where the left vertical fibration splits since the base now is  $(n+1)$ -connected. That means that  $f$  restricted to  $B\langle n+1 \rangle$  is null-homotopic. Since  $\text{map}_*(B\langle n+1 \rangle, B \text{aut}(F))_c$  is contractible (Proposition 2.1), applying the Zabrodsky Lemma, [13, Proposition 3.4], we deduce that  $f$  factors through  $K(\pi_{n+1}(B), n+1)$ . Therefore, to show that  $f$  is null-homotopic, we only need to prove that the induced map on homotopy groups  $\pi_{n+1}(B) \rightarrow \pi_{n+1}(B \text{aut}(F)) \cong \pi_n(F)$  is trivial. The naturality of the long exact sequence on homotopy groups shows that this morphism is the connecting morphism for the fibration  $F \rightarrow E \rightarrow B$ .

To prove (2) we use the fact that  $f$  factors actually through a map  $B \rightarrow B \text{aut}_*(F)$  if the fibration has a section. We first assume that  $B$  is  $n$ -connected. In this case, from the connectivity assumption on  $B$  and part (2) in Corollary 2.3 we see that  $f$  is null-homotopic. If  $B$  is  $(n-1)$ -connected, consider the fibration  $B\langle n \rangle \rightarrow B \rightarrow K(\pi_n(B), n)$ . The same argument as above shows that  $f$  factors through  $K(\pi_n(B), n)$ . Therefore,  $f$  is null-homotopic if and only if the induced map on homotopy groups  $\pi_n(B) \rightarrow \pi_n(B \text{aut}_*(F))$  is trivial.  $\blacksquare$

**Corollary 2.5.** *Let  $K(G, 1) \rightarrow E \rightarrow B$  be a fibration of connected spaces with a section where  $G$  is a discrete group. Then  $E \simeq K(G, 1) \times B$  if and only if the induced action  $\pi_1(B) \rightarrow \text{Aut}(G)$  is trivial.*

*Proof.* Note that  $\text{aut}_*(K(G, 1)) \simeq \text{Aut}(G)$ .  $\blacksquare$

**Corollary 2.6.** *Let  $n \geq 0$ ,  $A$  be a connected space, and  $B$  be an  $n$ -connected space such that  $\Omega^{n+1}B$  is  $A$ -local. Then the homotopy groups of  $\text{map}_*(A, B)_c$  are concentrated in degrees from 1 to  $n$  and  $\text{map}(A, B)_c \simeq \text{map}_*(A, B)_c \times B$ .*

*Proof.* Let us consider the evaluation fibration  $\text{map}_*(A, B)_c \rightarrow \text{map}(A, B)_c \rightarrow B$ . It has always a section, given by the constant maps. Since  $\Omega^{n+1}B$  is  $A$ -local, we see that  $\Omega^{n+1} \text{map}_*(A, B)_c$ , being weakly equivalent to  $\text{map}_*(A, \Omega^{n+1}B)$ , is contractible. Therefore the homotopy groups of  $\text{map}_*(A, B)_c$  in degree  $> n$  are all trivial. Part (2) of Theorem 2.4 applies.  $\blacksquare$

**Remark 2.7.** The sharpness of the connectivity assumption in Theorem 2.4 is illustrated by the following non-trivial fibrations. For part (1), consider the fibration  $K(\mathbb{Z}, n) \rightarrow S^{n+1}\langle n+1 \rangle \rightarrow S^{n+1}$ . The base space is only  $n$ -connected, and the fibration is not trivial, as it is classified by a non-trivial

$\text{map } S^{n+1} \rightarrow K(\mathbb{Z}, n+1)$ . In this example the connecting morphism  $\pi_{n+1}(S^{n+1}) \rightarrow \pi_n(K(\mathbb{Z}, n))$  is an isomorphism.

For part (2), let  $p$  be any prime,  $n \geq 2$  and  $F$  be the product  $K(\mathbb{Z}/p, 1) \times K(\mathbb{Z}/p, n)$ . Since  $F$  is an  $H$ -space, the identity component  $\text{map}_*(F, F)_{id}$  is weakly equivalent to the component of the constant map, i.e. is a product of Eilenberg-MacLane spaces, one of them being  $K(\mathbb{Z}/p, n-1)$ . Therefore  $\pi_n(B \text{aut}_*(F)) \cong \mathbb{Z}/p$  and there exists a map  $S^n \rightarrow B \text{aut}_*(F)$  classifying a split fibration over  $S^n$  with fiber  $F \simeq F[n]$ , which is not trivial. In this example  $\pi_n(S^n) \cong \mathbb{Z} \rightarrow \pi_n(B \text{aut}_*(F)) \cong \mathbb{Z}/p$  is the projection.

We apply now the results of this section to analyze certain mapping spaces. This allows us in particular to understand the space  $\text{map}(B\mathbb{Z}/p, BX)_c$  for any  $p$ -Noetherian group  $X$ .

**Proposition 2.8.** *Let  $Z$  be a space such that  $\Omega^2 Z$  is  $B\mathbb{Z}/p$ -local. Then the component of the mapping space  $\text{map}(B\mathbb{Z}/p, Z)_c$  splits as a product  $Z \times \text{map}_*(B\mathbb{Z}/p, Z)_c$  and  $\text{map}_*(B\mathbb{Z}/p, Z)_c$  is the classifying space of an elementary abelian  $p$ -group (not necessarily finite).*

*Proof.* By the work of Bousfield, [4, Theorem 7.2], the homotopy fiber of the nullification map  $Z \rightarrow P_{\Sigma B\mathbb{Z}/p} Z$  is a single Eilenberg-Mac Lane space  $K(P, 2)$ , where  $P$  is an abelian  $p$ -torsion group. He also shows that the fibration  $K(P, 2) \rightarrow Z \rightarrow P_{\Sigma B\mathbb{Z}/p} Z$  is principal. By adjunction, the component  $\text{map}_*(B\mathbb{Z}/p, P_{\Sigma B\mathbb{Z}/p} Z)_c$  is contractible. Therefore  $\text{map}_*(B\mathbb{Z}/p, Z)_c \simeq \text{map}_*(B\mathbb{Z}/p, K(P, 2))_c$ , which is the classifying space of the elementary abelian  $p$ -group  $W = \text{Hom}(\mathbb{Z}/p, P)$ .

Now, by Corollary 2.5, we only need to check that the action of  $\pi = \pi_1(Z)$  on  $W$  is trivial. By taking mapping spaces at the component of the constant map and evaluation, we obtain the following diagram of fibrations (since  $\text{map}_*(B\mathbb{Z}/p, P_{\Sigma B\mathbb{Z}/p} Z)_c$  is contractible):

$$\begin{array}{ccccc}
 BW & \xlongequal{\quad} & BW & & \\
 \downarrow & & \downarrow & & \\
 \text{map}(B\mathbb{Z}/p, K(P, 2))_c & \longrightarrow & \text{map}(B\mathbb{Z}/p, Z)_c & \longrightarrow & \text{map}(B\mathbb{Z}/p, P_{\Sigma B\mathbb{Z}/p} Z)_c \\
 \downarrow & & \downarrow & & \downarrow \simeq \\
 K(P, 2) & \longrightarrow & Z & \longrightarrow & P_{\Sigma B\mathbb{Z}/p} Z
 \end{array}$$

The bottom and middle horizontal fibrations are principal, therefore the action of the fundamental group of the base space,  $\pi_1 P_{\Sigma B\mathbb{Z}/p} Z \cong \pi$ , is trivial on all homotopy groups of the fiber, in particular on the fundamental group of the fiber. This action can be seen as conjugation in the fundamental group of the total space  $\text{map}(B\mathbb{Z}/p, Z)_c$ , but now it does not matter whether we look at the vertical fibration or the horizontal one (in both cases the induced morphism is surjective on the fundamental group). ■

**Corollary 2.9.** *Let  $X$  be a  $p$ -Noetherian group. Then the mapping space  $\mathrm{map}(B\mathbb{Z}/p, BX)_c$  splits as a product  $BX \times \mathrm{map}_*(B\mathbb{Z}/p, BX)_c$  where  $\mathrm{map}_*(B\mathbb{Z}/p, BX)_c$  is the classifying space of a finite elementary abelian  $p$ -group. In particular, the mapping space  $\mathrm{map}(B\mathbb{Z}/p, BX)_c$  is  $p$ -good,  $p$ -complete and  $H^*(\mathrm{map}(B\mathbb{Z}/p, BX)_c; \mathbb{F}_p)$  is of finite type.*

*Proof.* The finiteness of the elementary abelian  $p$ -group follows from Corollary 1.10. ■

In particular, we see that  $\mathrm{map}(B\mathbb{Z}/p, BX)_c$  is again the classifying space of a  $p$ -Noetherian group.

### 3. INDECOMPOSABLE ELEMENTS AND THE KRULL FILTRATION

As mentioned in the introduction, a good way to understand the cohomology of a space as an algebra over the Steenrod algebra is to look at the module of indecomposable elements  $QH^*(X; \mathbb{F}_p) = \tilde{H}^*(X; \mathbb{F}_p) / \tilde{H}^*(X; \mathbb{F}_p) \cdot \tilde{H}^*(X; \mathbb{F}_p)$ . An important observation here is that this definition depends on the choice of a base point, or more exactly on the choice of a component  $X_0$  if  $X$  is not connected. Since  $H^0(X; \mathbb{F}_p)$  is a  $p$ -Boolean algebra, it follows that  $QH^*(X; \mathbb{F}_p)$  is isomorphic to  $QH^*(X_0; \mathbb{F}_p)$ .

There is a (Krull) filtration of the category  $\mathcal{U}$  of unstable modules,  $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \dots$  such that  $\mathcal{U}_0$  consists in the locally finite unstable module. Schwartz established in [35, Theorem 6.2.4] a criterion to check whether (and where) an unstable module lives in the Krull filtration, namely  $M \in \mathcal{U}_n$  if and only if  $\bar{T}^{n+1}M = 0$ , where  $\bar{T}$  is Lannes' reduced  $T$ -functor.

Therefore our objective in this section is to prove that  $\bar{T}^2 QH^*(BX; \mathbb{F}_p) = 0$  for a  $p$ -Noetherian group. To do so, we need first to find a geometrical interpretation of the reduced  $T$ -functor.

Recall that “under some mild assumptions”,  $TH^*(Z; \mathbb{F}_p) \cong H^*(\mathrm{map}(B\mathbb{Z}/p, Z); \mathbb{F}_p)$ . Lannes' standard mild assumptions on  $Z$  are that  $TH^*(Z; \mathbb{F}_p)$  is of finite type (or  $H^*(\mathrm{map}(B\mathbb{Z}/p, Z); \mathbb{F}_p)$  is of finite type), and that  $\mathrm{map}(B\mathbb{Z}/p, Z)$  is  $p$ -good, [28, Proposition 3.4.4]. We will not need to understand globally the mapping space, but restrict our attention to the component  $\mathrm{map}(B\mathbb{Z}/p, Z)_c$  of the constant map, the natural choice of base point in the full mapping space. We thus only consider the component  $T_c(H^*(Z; \mathbb{F}_p))$  of Lannes'  $T$ -functor.

**Theorem 3.1.** *Let  $Z$  be a  $p$ -complete space such that  $H^*(Z; \mathbb{F}_p)$  and  $H^*(\mathrm{map}(B\mathbb{Z}/p, Z)_c)$  are of finite type. If  $\Omega^2 Z$  is  $B\mathbb{Z}/p$ -local, then*

$$\bar{T}QH^*(Z; \mathbb{F}_p) \cong QH^*(\mathrm{map}_*(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p).$$

*In particular the unstable module  $QH^*(Z; \mathbb{F}_p)$  lies in  $\mathcal{U}_1$ .*

*Proof.* In Proposition 2.8 we obtained a splitting  $\mathrm{map}(B\mathbb{Z}/p, Z)_c \simeq \mathrm{map}_*(B\mathbb{Z}/p, Z)_c \times Z$  and an equivalence  $\mathrm{map}_*(B\mathbb{Z}/p, Z)_c \simeq BW$  where  $W$  is an elementary abelian  $p$ -group. With the hypothesis of the theorem this splitting shows that  $H^*(BW; \mathbb{F}_p)$  is of finite type and therefore  $W$  is finite. Therefore  $\mathrm{map}(B\mathbb{Z}/p, Z)_c$  is  $p$ -good. Since moreover  $H^*(\mathrm{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$  is of

finite type by assumption, we can apply Lannes' result [28, Proposition 3.4.4] and deduce that the  $T$ -functor computes what it should:  $T_c H^*(Z; \mathbb{F}_p) \cong H^*(\text{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$ .

Notice also that  $QH^*(\text{map}(B\mathbb{Z}/p, Z); \mathbb{F}_p) \cong QH^*(\text{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$ . Since Lannes'  $T$ -functor commutes with taking the module of indecomposable elements,  $TQH^*(Z; \mathbb{F}_p) \cong QTH^*(Z; \mathbb{F}_p)$ . But in degree zero  $TH^*(Z; \mathbb{F}_p)$  is a Boolean algebra, [35, Section 3.8], so that  $QTH^*(Z; \mathbb{F}_p) \cong Q(T_c H^*(Z; \mathbb{F}_p))$ , which is isomorphic to  $QH^*(\text{map}(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$ . The splitting yields next an isomorphism

$$TQH^*(Z; \mathbb{F}_p) \cong QH^*(\text{map}_*(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p) \oplus QH^*(Z; \mathbb{F}_p)$$

so that we have finally identified  $\bar{T}QH^*(Z; \mathbb{F}_p) \cong QH^*(\text{map}_*(B\mathbb{Z}/p, Z)_c; \mathbb{F}_p)$ . This proves the first part of the theorem. For the second claim, use the fact that  $\text{map}_*(B\mathbb{Z}/p, Z)_c \simeq BW$ , the classifying space of a finite elementary abelian group. The cohomology of  $W$  is finitely generated as an algebra, so  $QH^*(BW; \mathbb{F}_p)$  is finite and lies in  $\mathcal{U}_0$ . Therefore  $\bar{T}QH^*(BW; \mathbb{F}_p) = 0$ , or equivalently  $\bar{T}^2QH^*(Z; \mathbb{F}_p) = 0$ , and so  $QH^*(Z; \mathbb{F}_p)$  lies in  $\mathcal{U}_1$ .  $\blacksquare$

Let us now turn to an even finer analysis of the module of indecomposable elements. Let us denote by  $Q_1$  the unstable module  $QH^*(B\mathbb{Z}/p; \mathbb{F}_p)$  of the cohomology of a cyclic group of order  $p$ . At the prime  $p = 2$ , the unstable module  $Q_1$  is isomorphic to  $\Sigma\mathbb{F}_2 = \Sigma F(0)$ . At an odd prime  $Q_1$  is an unstable module with one generator  $t$  in degree 1 and its Bockstein  $\beta t$  in degree 2.

**Proposition 3.2.** *Let  $Z$  be a  $p$ -complete space such that  $H^*(Z; \mathbb{F}_p)$  and  $H^*(\text{map}(B\mathbb{Z}/p, Z)_c)$  are of finite type. Assume that  $\Omega^2 Z$  is  $B\mathbb{Z}/p$ -local. Define  $Q_1 = QH^*(B\mathbb{Z}/p; \mathbb{F}_p)$ . Then there exists a morphism  $QH^*(Z; \mathbb{F}_p) \rightarrow F(1) \otimes (Q_1^{\oplus k})$  with finite cokernel and locally finite kernel.*

*Proof.* Schwartz characterizes in [37, Proposition 2.3] the unstable modules  $M$  in  $\mathcal{U}_1$  as those sitting in an exact sequence  $0 \rightarrow K \rightarrow M \rightarrow F(1) \otimes L \rightarrow N \rightarrow 0$ , where  $K, L$ , and  $N$  are locally finite (i.e. in  $\mathcal{U}_0$ ). In particular  $\bar{T}M \cong L$  since  $\bar{T}F(1) = F(0)$  and  $T$  commutes with tensor products, [35, Theorem 3.5.1]. In our case we know from the previous theorem that  $\bar{T}QH^*(Z; \mathbb{F}_p) \cong QH^*(BW; \mathbb{F}_p)$  where  $W$  is an abelian elementary group, say of rank  $k$ . Thus  $L = Q_1^{\oplus k}$ . The quotient  $N$  of  $F(1) \otimes (Q_1^{\oplus k})$  will be finitely generated. As it is locally finite it must be finite.  $\blacksquare$

We finally come back to  $p$ -Noetherian groups and prove that the module of indecomposable elements  $QH^*(BX; \mathbb{F}_p)$  is as small as expected.

**Theorem 3.3.** *Let  $X$  be a  $p$ -Noetherian group. Then*

$$\bar{T}QH^*(BX; \mathbb{F}_p) \cong QH^*(\text{map}_*(B\mathbb{Z}/p, BX)_c; \mathbb{F}_p).$$

*In particular the unstable module  $QH^*(BX; \mathbb{F}_p)$  lies in  $\mathcal{U}_1$ .*

*Proof.* The assumptions in Theorem 3.1 are satisfied by Corollary 2.9.  $\blacksquare$

## 4. FIBRATIONS OVER SPACES WITH FINITE COHOMOLOGY

In our study of  $H^*(BX; \mathbb{F}_p)$ , we have already managed to prove that  $QH^*(BX; \mathbb{F}_p)$  lives in  $\mathcal{U}_1$ , that is only one stage higher than where  $QH^*(X; \mathbb{F}_p)$  lives. What is left to prove is that  $H^*(BX; \mathbb{F}_p)$  is finitely generated as an algebra over the Steenrod algebra. Therefore we analyze the fibration  $K(P, 2)_p^\wedge \rightarrow BX \rightarrow BY$  of Theorem 1.9.

Let  $F \rightarrow E \rightarrow B$  be a fibration where both  $H^*(B; \mathbb{F}_p)$  and  $H^*(F; \mathbb{F}_p)$  are finitely generated  $\mathcal{A}_p$ -algebras. In this situation, we ask whether the same finiteness condition holds for  $H^*(E; \mathbb{F}_p)$ . When the fibration is one of  $H$ -spaces and  $H$ -maps we proved in [9] that this is true. But in general some restrictions have to be imposed, even when the fiber is a single Eilenberg-Mac Lane space as shown by the following example.

**Example 4.1.** Consider the folding map  $K(\mathbb{Z}, 3) \vee K(\mathbb{Z}, 3) \rightarrow K(\mathbb{Z}, 3)$ . An easy application of Puppe's theorem [34], shows that the homotopy fiber is  $\Sigma\Omega K(\mathbb{Z}, 3) \simeq \Sigma K(\mathbb{Z}, 2)$ . Therefore there exists a fibration

$$K(\mathbb{Z}, 2) \rightarrow \Sigma K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}, 3) \vee K(\mathbb{Z}, 3).$$

The mod 2 cohomology of the fiber is finitely generated as an algebra, the cohomology of the base space is generated over  $\mathcal{A}_2$  by the two fundamental classes in degree 3. However the cohomology of  $\Sigma K(\mathbb{Z}, 2)$  is not finitely generated over  $\mathcal{A}_2$  (as it is a suspension it would be finitely generated as an unstable module, and therefore would belong to some stage of the Krull filtration; by Schwartz's solution [36] to Kuhn's non-realizability conjecture this would imply that the cohomology were locally finite, which it is not).

This example indicates that we must impose stronger conditions on the base space of the fibration to make sure that the cohomology of the total space is finitely generated as an algebra over  $\mathcal{A}_p$ . In this section we study fibrations  $F \rightarrow E \rightarrow B$  where  $\pi_1 B$  acts trivially on the cohomology of the fiber. We will assume that  $H^*(B; \mathbb{F}_p)$  is *finite* and the fiber  $F$  is a finite product of Eilenberg Mac-Lane spaces  $\prod_{i=1}^q K(A_i, n_i)$  where  $A_i$  is a finitely generated abelian group for all  $i$ . Both assumptions will play an essential role in the analysis of the cohomology of the total space. The finiteness of the base forces the Serre spectral sequence to collapse at some finite stage and the hypothesis on  $A_i$  implies that the cohomology of  $K(A_i, n_i)$  is generated, as an algebra over the Steenrod algebra  $\mathcal{A}_p$ , by a finite number of fundamental classes  $\iota_1, \dots, \iota_m$  of degree  $n$ , and possibly certain higher Bockstein on these classes. It is a free algebra by work of Serre at the prime 2, [38], and Cartan at odd primes, [7].

**Lemma 4.2.** *There exists a splitting  $H^*(\prod K(A_i, n_i); \mathbb{F}_p) \cong F^* \otimes G^*$  of algebras where  $F^*$  is finitely generated as an algebra, and  $G^*$  consists of permanent cycles in the Serre spectral sequence. Moreover  $G^*$  is finitely generated as an algebra over  $\mathcal{A}_p$ .*

*Proof.* By Kudo's transgression theorem, all classes obtained by applying Steenrod operations to transgressive operations are transgressive. Let us choose therefore an integer  $r$  larger than the dimension of the cohomology of the base. If  $\{x_1, \dots, x_k\}$  is a set of generators of  $H^*(F; \mathbb{F}_p)$  as an  $\mathcal{A}_p$ -algebra, the elements  $1 \otimes \mathcal{P}^I x_k$  are permanent cycles for any sequence  $I$  of degree larger than  $r - n - 1$  and any  $k$ .

We will say that such generators  $\mathcal{P}^I x_k$  have *large* degree and the others, of which there is only a finite number, have small degree. We define now  $F^*$  to be the subalgebra generated by the generators of small degree and  $G^*$  by all other large degree generators. Then  $H^*(F; \mathbb{F}_p) \cong F^* \otimes G^*$ .

The last claim is proven by looking at the inclusion of algebras  $G^* \subset H^*(F; \mathbb{F}_p)$ . At the level of modules of indecomposable elements it induces an inclusion  $QG^* \subset QH^*(F; \mathbb{F}_p)$ , because of the freeness of  $H^*(F; \mathbb{F}_p)$  and our choices of generators. Since the category  $\mathcal{U}$  of unstable modules is locally Noetherian, [35, Theorem 1.8.1], the unstable module  $QG^*$  is finitely generated. Therefore,  $G^*$  is a finitely generated  $\mathcal{A}_p$ -algebra. ■

The proof of the next proposition follows the line of the Dwyer-Wilkerson result [18, Proposition 12.4], see also Evens, [19].

**Proposition 4.3.** *The cohomology of the total space  $H^*(E; \mathbb{F}_p)$  is finitely generated as a module over  $H^*(B; \mathbb{F}_p)[z_1, \dots, z_k] \otimes G^*$ .*

*Proof.* The free algebra  $F^*$  is finitely generated and we consider first all polynomial generators  $a_1, \dots, a_k$ . We define  $z_i = (a_i)^{p^{n_i}}$  where  $n_i$  is the smallest integer such that this power of  $a_i$  is a permanent cycle (it exists since these powers are transgressive, compare with the proof of Lemma 4.2). Then  $F^*$  is a finitely generated module over  $\mathbb{F}_p[z_1, \dots, z_k]$ . Chose now a finite set of generators  $g_1, \dots, g_r$  of  $G^*$  as an algebra over the Steenrod algebra.

The elements  $z_i$  and the elements  $g_1, \dots, g_r$  are permanent cycles in the vertical axis of the Serre spectral sequence, one can thus choose elements  $z'_i$  and  $g'_j$  in  $H^*(E; \mathbb{F}_p)$  whose images in  $H^*(K(A, n); \mathbb{F}_p)$  are the  $z_i$ 's and the  $g'_j$ 's. Better said, since both  $\mathbb{F}_p[z_i]$  and  $G^*$  are free algebras, we choose an algebra map  $s: \mathbb{F}_p[z_i] \otimes G^* \hookrightarrow H^*(E; \mathbb{F}_p)$ . The elements in  $H^*(B; \mathbb{F}_p)$  act on  $H^*(E; \mathbb{F}_p)$  via  $p^*: H^*(X; \mathbb{F}_p) \rightarrow H^*(E; \mathbb{F}_p)$ . This explains the module structure.

We see that  $E_\infty = E_r$  is finitely generated as a module over  $H^*(B; \mathbb{F}_p)[z_1, \dots, z_k] \otimes G^*$ . Therefore so is  $H^*(E; \mathbb{F}_p)$  by [41, Corollary VII.3.3]. ■

The difficulty to infer information about the  $\mathcal{A}_p$ -algebra structure from the module structure is that the algebra map  $s$  is not a map of  $\mathcal{A}_p$ -algebras. To circumvent this problem we will appeal to the algebraic result proved in the appendix A.

**Theorem 4.4.** *Consider a fibration  $\prod_{i=1}^q K(A_i, n_i) \rightarrow E \rightarrow B$  where  $H^*(B; \mathbb{F}_p)$  is finite and  $A_i$  is a finitely generated abelian group for all  $i$ . The cohomology  $H^*(E; \mathbb{F}_p)$  is then finitely generated as an algebra over  $\mathcal{A}_p$ .*

*Proof.* In the notation of the appendix,  $H^*(B; \mathbb{F}_p)[z_1, \dots, z_k]$  is the connected and commutative finitely generated algebra  $C^*$ , and  $B^* = H^*(E; \mathbb{F}_p)$ . By Proposition 4.3,  $H^*(E; \mathbb{F}_p)$  is a finitely generated  $C^* \otimes G^*$ -module. The action of  $C^* \otimes G^*$  on  $B^*$  has been defined in the previous proof via an algebra map (constructed from a section  $s : G^* \rightarrow B^*$ ), thus  $B^*$  is a  $C^* \otimes G^*$ -algebra. Define now  $\pi : H^*(\prod K(A_i, n_i); \mathbb{F}_p) \cong F^* \otimes G^* \rightarrow G^*$  to be the projection and  $p : B^* \rightarrow G^*$  to be the composite  $\pi \circ i^*$ . This is a morphism of  $G^*$ -modules so that Proposition A.2 applies. Hence  $H^*(E; \mathbb{F}_p)$  is finitely generated as an algebra over  $\mathcal{A}_p$ .  $\blacksquare$

**Remark 4.5.** The nature of Theorem 4.4 is purely cohomological. Therefore the same statement remains true if we relax the assumption on the fiber in the following way: The fiber  $F$  should be homotopic, up to  $p$ -completion, to  $\prod_{i=1}^q K(A_i, n_i)$  where each  $A_i$  is a finitely generated abelian group. This will allow us to include summands of the form  $\mathbb{Z}_{p^\infty}$  or  $\mathbb{Z}_p^\wedge$ . In fact the same proof goes through with the assumption that  $H^*(F; \mathbb{F}_p)$  is a free algebra, finitely generated as an algebra over  $\mathcal{A}_p$ .

As a byproduct we obtain the following result on highly connected covers of finite complexes. For a mod  $p$  finite  $H$ -space  $B$  we proved in [9] that the mod  $p$  cohomology of  $B\langle n \rangle$  is finitely generated for *any* integer  $n$ .

**Corollary 4.6.** *Let  $n \geq 2$  and  $B$  be an  $(n-1)$ -connected space with finite mod  $p$  cohomology. Then  $H^*(B\langle n \rangle; \mathbb{F}_p)$  is finitely generated as an algebra over  $\mathcal{A}_p$ . Moreover the unstable module of indecomposable elements  $QH^*(B\langle n \rangle; \mathbb{F}_p)$  lies in  $\mathcal{U}_{n-2}$ .*

*Proof.* The cohomology  $H^*(B\langle n \rangle; \mathbb{F}_p)$  is finitely generated as an algebra over  $\mathcal{A}_p$  by a direct application of Theorem 4.4. Now from Corollary 2.6 we infer that  $\text{map}(B\mathbb{Z}/p, B\langle n \rangle)_c$  splits as a product  $B\langle n \rangle \times \text{map}_*(B\mathbb{Z}/p, B\langle n \rangle)_c$ . Since  $B$  itself is  $B\mathbb{Z}/p$ -local by Miller's Theorem, [30], this pointed mapping space is equivalent to  $\text{map}_*(B\mathbb{Z}/p, K(\pi_n X, n-1)_c)$ . This is a product of Eilenberg-Mac Lane spaces, the highest of them being  $K(A, n-2)$  where  $A = \text{Hom}(\mathbb{Z}/p, \pi_n B)$ . The module of indecomposable elements of its cohomology lies in  $\mathcal{U}_{n-3}$ . Thus  $QH^*(B\langle n \rangle; \mathbb{F}_p)$  lies in  $\mathcal{U}_{n-2}$ , the proof is analogous to that of Theorem 3.1.  $\blacksquare$

## 5. THE COHOMOLOGY OF $p$ -NOETHERIAN GROUPS

We are about to conclude our study of the cohomology of classifying spaces of  $p$ -Noetherian groups. We have seen that any  $p$ -Noetherian group is the total space of a fibration over a  $p$ -compact group with fiber an Eilenberg-Mac Lane space. Recall from [18]'s main theorem that the mod  $p$ -cohomology of the classifying space of a  $p$ -compact group is finitely generated as an algebra. Our objective is to prove the following theorem, which together with Theorem 3.3 gives a very accurate description for the cohomology of classifying spaces of  $p$ -Noetherian groups.



**Theorem 5.1.** *Let  $(X, BX, e)$  be a  $p$ -Noetherian group. Then  $H^*(BX; \mathbb{F}_p)$  is finitely generated as an algebra over the Steenrod algebra.*

The strategy is to use the fibration  $K(P, 2)_p^\wedge \rightarrow BX \rightarrow BY$  of Theorem 1.9. As we do not know whether the Serre spectral sequence collapses at some finite stage, we reduce the problem in several steps to the study of a spectral sequence over a finite base (in order to apply our results from the previous section).

A  $p$ -compact toral group  $P$  is a  $p$ -compact group which is an extension of a  $p$ -compact torus  $((S^1)^n)_p^\wedge$  by a finite  $p$ -group. Dwyer and Wilkerson in [18] show that any  $p$ -compact group  $Y$  admits a maximal  $p$ -compact toral subgroup  $N \leq Y$  such that the homotopy fiber  $Y/N$  of the map  $Bi: BN \rightarrow BY$  has finite mod  $p$  cohomology and Euler characteristic prime to  $p$  (see [18, Proof of 2.3]). A transfer argument (see [18, Theorem 9.13]) then shows that  $Bi$  induces a monomorphism in mod  $p$  cohomology. Consider now the pullback diagram

$$\begin{array}{ccc} \widetilde{BN} & \xrightarrow{\bar{Bi}} & BX \\ \downarrow & & \downarrow \\ BN & \xrightarrow{Bi} & BY. \end{array}$$

First note that, by Proposition 1.8,  $\widetilde{BN}$  is the classifying space of a  $p$ -Noetherian group because we have a fibration  $K(P, 2)_p^\wedge \rightarrow \widetilde{BN} \rightarrow BN$ . We will first show that  $H^*(\widetilde{BN}; \mathbb{F}_p)$  is a finitely generated  $\mathcal{A}_p$ -algebra by using this fibration, and then we will show that the cohomology  $H^*(BX; \mathbb{F}_p)$  is a finitely generated  $\mathcal{A}_p$ -algebra by using the fibration  $Y/N \rightarrow \widetilde{BN} \rightarrow BX$ . We start with a technical result which will be used in both steps of the proof.

**Proposition 5.2.** *Let  $F \rightarrow E \xrightarrow{q} B$  be a fibration such that  $\pi_1(B)$  acts nilpotently on  $H_*(F; \mathbb{F}_p)$ . Assume that  $q$  induces an isomorphism  $\bar{T}QH^*B \cong \bar{T}QH^*E$ ,  $H^*(F; \mathbb{F}_p)$  is locally finite, and  $H^*(E; \mathbb{F}_p)$  is a finitely generated  $\mathcal{A}_p$ -algebra. Then, if  $\text{Ker } q^*$  is a finitely generated ideal then  $H^*(B; \mathbb{F}_p)$  is a finitely generated  $\mathcal{A}_p$ -algebra.*

*Proof.* Let  $A$  be the algebra which is the quotient of  $H^*(E; \mathbb{F}_p)$  by the ideal generated by the image of  $q^*$ , that is,  $\mathbb{F}_p \otimes_{H^*(B; \mathbb{F}_p)} H^*(E; \mathbb{F}_p)$ . There is a coexact sequence

$$H^*(B; \mathbb{F}_p) // \text{Ker } q^* \xrightarrow{q^*} H^*(E; \mathbb{F}_p) \rightarrow A.$$

Since  $H^*(E; \mathbb{F}_p)$  is a finitely generated  $\mathcal{A}_p$ -algebra, the same is true for  $A$ . Therefore  $QA$  is a finitely generated  $\mathcal{A}_p$ -module, which is the the cokernel of  $Qq^*$  by right-exactness of the functor  $Q$ . Moreover, since by assumption  $\bar{T}QH^*B \cong \bar{T}QH^*E$ , it follows from exactness of the reduced  $T$  functor that  $\bar{T}QA = 0$ . This means that  $QA$  belongs to  $\mathcal{U}_0$ , i.e. it is locally finite, hence finite. Equivalently  $A$  is a finitely generated algebra.

We want to show that  $A$  is in fact a finite algebra. The fiber inclusion  $F \rightarrow E$  of the fibration  $q$  induces a morphism  $\iota^*: A \rightarrow H^*(F; \mathbb{F}_p)$ . A careful study of the Eilenberg-Moore spectral sequence, [29, Theorem 0.5] (for the prime 2) or [35, Theorem 8.7.8], shows that this morphism is an  $F$ -monomorphism. Take now any element  $a \in A$ . Because  $H^*(F; \mathbb{F}_p)$  is locally finite, there exists  $M > 0$  such that  $a^{p^M} \in \text{Ker } \iota^*$ , which is nilpotent. There exists thus  $N > 0$  such that  $a^{p^{N+M}} = 0$ . Since all elements of  $A$  are nilpotent and it is a finitely generated algebra,  $A$  must be finite.

For the last statement, note that, as an  $H^*(B; \mathbb{F}_p)$ -module, the cohomology  $H^*(E; \mathbb{F}_p)$  is isomorphic to  $(H^*(B; \mathbb{F}_p) // \text{Ker } q^*) \{b_1, \dots, b_k\}$ , where the  $b_i$ 's are the generators of  $A$  as a (graded)  $\mathbb{F}_p$ -vector space. This description shows that the morphism  $Q(H^*(B; \mathbb{F}_p) // \text{Ker } q^*) \rightarrow QH^*(E; \mathbb{F}_p)$  is an isomorphism in high degrees. That is, there exists  $K > 0$  such that

$$(Q(H^*(B; \mathbb{F}_p) // \text{Ker } q^*))^{>K} \cong (QH^*(E; \mathbb{F}_p))^{>K},$$

which is an unstable submodule of  $QH^*(E; \mathbb{F}_p)$ . Since  $H^*(E; \mathbb{F}_p)$  is a finitely generated  $\mathcal{A}_p$ -algebra,  $QH^*(E; \mathbb{F}_p)$  is a finitely generated  $\mathcal{A}_p$ -module. But then, as the category of unstable modules over the Steenrod algebra is locally Noetherian, [35, Theorem 1.8.1], the same holds for  $(Q(H^*(B; \mathbb{F}_p) // \text{Ker } q^*))^{>K}$ . In particular,  $Q(H^*(B; \mathbb{F}_p) // \text{Ker } q^*)$  is a finitely generated  $\mathcal{A}_p$ -module since it is of finite type, that is  $H^*(B; \mathbb{F}_p) // \text{Ker } q^*$  is a finitely generated  $\mathcal{A}_p$ -algebra.

Finally,  $H^*(B; \mathbb{F}_p)$  is generated by  $H^*(B; \mathbb{F}_p) // \text{Ker } q^*$  and  $\text{Ker } q^*$ , it is therefore also finitely generated as an algebra over the Steenrod algebra since  $\text{Ker } q^*$  is a finitely generated ideal.  $\blacksquare$

We apply next this result to the fibration  $Y/N \rightarrow \widetilde{BN} \rightarrow BX$ .

**Corollary 5.3.**  *$H^*(BX; \mathbb{F}_p)$  is a finitely generated algebra if  $H^*(\widetilde{BN}; \mathbb{F}_p)$  is so.*

*Proof.* Consider the fibration  $Y/N \rightarrow \widetilde{BN} \xrightarrow{B\tilde{i}} BX$  and the diagram of horizontal fibrations

$$\begin{array}{ccccc} \text{map}(B\mathbb{Z}/p, Y/N)_c & \longrightarrow & \text{map}(B\mathbb{Z}/p, \widetilde{BN})_c & \longrightarrow & \text{map}(B\mathbb{Z}/p, BX)_c \\ \simeq \downarrow \text{ev} & & \downarrow \text{ev}_r & & \downarrow \text{ev}_n \\ Y/N & \longrightarrow & \widetilde{BN} & \longrightarrow & BX \end{array}$$

where the left vertical arrow is an equivalence since the fiber  $Y/N$  is mod  $p$  finite. Therefore  $B\tilde{i}$  induces an equivalence  $\text{map}_*(B\mathbb{Z}/p, \widetilde{BN})_c \rightarrow \text{map}_*(B\mathbb{Z}/p, BX)_c$ . Both  $BX$  and  $\widetilde{BN}$  are  $p$ -Noetherian groups and Theorem 3.3 implies then that  $\bar{T}QH^*BX \rightarrow \bar{T}QH^*\widetilde{BN}$  is an isomorphism.

We conclude now by Proposition 5.2, because  $\pi_1(BX)$  is a finite  $p$ -group and  $q^*$  is injective by a transfer argument [18, Theorem 9.13] (the Euler characteristic  $\chi(Y/N)$  is prime to  $p$ ).  $\blacksquare$

The key fact in the next argument is that any  $p$ -compact toral group satisfies the Peter-Weyl theorem. That is, it admits a homotopy monomorphism into  $U(n)_p^\wedge$  for some  $n$ . This is shown for example in [26, Proposition 2.2]. Let us choose then such a map  $\rho: BN \rightarrow BU(n)_p^\wedge$  for the maximal  $p$ -compact toral group  $BN$ . The mod  $p$  cohomology of the fibre  $F$  is hence  $\mathbb{F}_p$ -finite.

**Proposition 5.4.** *Let  $E_0$  be the homotopy pull-back of the diagram  $F \rightarrow BN \leftarrow \widetilde{BN}$ . Then  $H^*(E_0; \mathbb{F}_p)$  is finitely generated as an algebra over the Steenrod algebra.*

*Proof.* The space  $E_0$  fits by construction into a fibration  $K(P, 2)_p^\wedge \rightarrow E_0 \rightarrow F$ . Since the mod  $p$  cohomology of  $F$  is finite, Theorem 4.4 shows that  $H^*(E_0; \mathbb{F}_p)$  is a finitely generated  $\mathcal{A}_p$ -algebra.  $\blacksquare$

Let us now concentrate on the map  $E_0 \rightarrow \widetilde{BN}$ . We will filter it by using the top left block diagonal inclusions  $U(r-1) \subset U(r)$  for  $1 \leq r \leq n$ . At the level of classifying spaces these inclusions induce oriented spherical fibrations

$$S^{2r-1} \rightarrow BU(r-1) \rightarrow BU(r).$$

Let  $BU(r) \rightarrow BU(n)$  be the appropriate composite and define then  $E_r$  to be the homotopy pullback of  $\widetilde{BN} \xrightarrow{p} BU(n)_p^\wedge \leftarrow BU(r)_p^\wedge$ . Thus  $E_n = \widetilde{BN}$  and, for  $1 \leq r \leq n$ , we have spherical fibrations  $(S^{2r-1})_p^\wedge \rightarrow E_{r-1} \rightarrow E_r$ . We summarize next some important properties of these spaces.

**Proposition 5.5.** *The component  $\text{map}(B\mathbb{Z}/p, E_r)_c$  splits as a product  $E_r \times \text{map}_*(B\mathbb{Z}/p, E_r)_c$  for any  $0 \leq r \leq n$ .*

*Proof.* Let us denote by  $ev_r: \text{map}(B\mathbb{Z}/p, E_r)_c \rightarrow E_r$  the evaluation at the component of the constant map and consider the following commutative square of horizontal fibrations of connected spaces

$$\begin{array}{ccccc} \text{map}(B\mathbb{Z}/p, H)_c & \longrightarrow & \text{map}(B\mathbb{Z}/p, E_r)_c & \longrightarrow & \text{map}(B\mathbb{Z}/p, \widetilde{BN})_c \\ \simeq \downarrow ev & & \downarrow ev_r & & \downarrow ev_n \\ H & \longrightarrow & E_r & \longrightarrow & \widetilde{BN} \end{array}$$

The homotopy fiber  $H$  has finite cohomology, and is thus equivalent via the evaluation map to the homotopy fiber  $\text{map}(B\mathbb{Z}/p, H)_c$  of the top map. This shows that the right hand square is a pull-back square. But  $ev_n$  is a trivial fibration by Proposition 2.9. Hence so is  $ev_r$  and  $\text{map}(B\mathbb{Z}/p, E_r)_c$  must split as a product  $E_r \times \text{map}_*(B\mathbb{Z}/p, E_r)_c$ .  $\blacksquare$

**Proposition 5.6.** *The morphism of  $\mathcal{A}_p$ -modules  $\bar{T}QH^*(\widetilde{BN}; \mathbb{F}_p) \rightarrow \bar{T}QH^*(E_r; \mathbb{F}_p)$  induced by  $E_r \rightarrow \widetilde{BN}$  is an isomorphism.*

*Proof.* We have seen in the previous proposition that the map  $E_r \rightarrow BX$  induces a weak equivalence on the connected component of the constant map in the pointed mapping space  $\text{map}_*(B\mathbb{Z}/p, -)_c$ . From this point on, the same argument as in the proof of Theorem 3.3 goes through.  $\blacksquare$

*Proof of Theorem 5.1.* We know that  $H^*(E_0; \mathbb{F}_p)$  is finitely generated as an algebra over the Steenrod algebra. It is thus sufficient to prove that  $H^*(E_r; \mathbb{F}_p)$  is a finitely generated  $\mathcal{A}_p$ -algebra if so is  $H^*(E_{r-1}; \mathbb{F}_p)$ . Denote by  $q_r$  the map  $E_{r-1} \rightarrow E_r$  turned into a fibration. Since  $\text{Ker } q_r^*$  is generated by a single element, namely the Euler class, Proposition 5.2 applies.  $\blacksquare$

**Remark 5.7.** In general, an unstable algebra which is finitely generated as an algebra over  $\mathcal{A}_p$  may contain unstable subalgebras which are not finitely generated over  $\mathcal{A}_p$ . Such an example appears in [9, Remark 2.2] as an unstable subalgebra  $B$  of  $H^*(BS^1 \times S^2; \mathbb{F}_p) \cong \mathbb{F}_p[x] \otimes E(y)$ , which is not a finitely generated  $B$ -module.  $B$  is the ideal generated by  $y$  turned into an unstable algebra by adding 1. The quotient is a polynomial algebra, in particular it is not finite.

**Remark 5.8.** To prove Theorem 5.1 one could also use the fact that  $p$ -compact groups satisfy the Peter-Weyl theorem (see [3, Theorem 1.6] and [2, Remark 7.3]). This would slightly shorten the proof and avoid the use of the maximal  $p$ -compact toral subgroup. But the Peter-Weyl theorem for  $p$ -compact groups is proved using the classification of  $p$ -compact groups and we want to emphasize that Theorem 5.1 does not depend on the classification.

**Corollary 5.9.** *Let  $G$  be a simply connected, simple, compact Lie group. Then  $H^*(BG\langle 4 \rangle; \mathbb{F}_p)$  is finitely generated as an algebra over the Steenrod algebra and  $QH^*(BG\langle 4 \rangle; \mathbb{F}_p)$  belongs to  $\mathcal{U}_1$ . In fact there exists a morphism  $QH^*(BG\langle 4 \rangle; \mathbb{F}_p) \rightarrow F(1) \otimes Q_1$  with finite kernel and cokernel.*

*Proof.* Recall that  $Q_1 = QH^*(B\mathbb{Z}/p; \mathbb{F}_p)$ . Since  $BG$  is 3-connected and  $\pi_4 BG \cong \mathbb{Z}$ , Proposition 3.2 yields a morphism  $QH^*(BG\langle 4 \rangle; \mathbb{F}_p) \rightarrow F(1) \otimes Q_1$  (the elementary abelian group  $W$  appearing there has rank one). The cokernel is finite and the kernel locally finite. But since  $QH^*(BG\langle 4 \rangle; \mathbb{F}_p)$  is a finitely generated module over  $\mathcal{A}_p$ , the kernel must be finite. ■

**Example 5.10.** Let  $X$  be either any simply connected compact Lie group (such as  $Spin(10)$ , which is one of the smallest examples Harada and Kono could not handle at the prime 2) or one of the  $p$ -compact groups number 2b, 23 or 30 in the Shephard-Todd list. For odd primes, the exotic  $p$ -compact groups arising from this construction are again non-modular. Hence, in all the cases,  $H^*(X; \mathbb{Z}_p^\wedge)$  is torsion free. The mod  $p$  cohomology is given by  $H^*(BX_{23}; \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{12}, x_{20}]$ ,  $H^*(BX_{30}; \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{24}, x_{40}, x_{60}]$ , and  $H^*(BX_{2b,m}; \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2m}]$ . Since all examples are torsion free, the same techniques used by Harada and Kono in [24] and [23] show that  $H^*(BX\langle 4 \rangle; \mathbb{F}_p) \cong H^*(BX; \mathbb{F}_p)/J \otimes R_h$  where  $J$  is the ideal generated by  $x_4, \mathcal{P}^1 x_4, \dots, \mathcal{P}^h x_4$  for a certain  $h$ , and  $R_h$  is an unstable subalgebra of  $H^*(K(\mathbb{Z}_p^\wedge, 3); \mathbb{F}_p)$  which is finitely generated over  $\mathcal{A}_p$ . In fact Theorems 5.1 and 3.3 show directly that  $QH^*(BX\langle 4 \rangle; \mathbb{F}_p)$  is finitely generated as a module over  $\mathcal{A}_p$  and belongs to  $\mathcal{U}_1$ . From the last corollary we see for example that  $QH^*(BSpin(10)\langle 4 \rangle; \mathbb{F}_2)$  differs from  $\Sigma F(1)$  in only a finite number of “low dimensional” classes.

Our techniques also allow us to say something about the cohomology of some higher connected covers.

**Example 5.11.** Let  $X$  be any exotic  $p$ -compact groups except those corresponding to number 2b, 23 or 30 in the Shephard-Todd list. They are  $(n-1)$ -connected for some integer  $n > 4$  (for example at the prime 2 the only exotic example is  $BDI(4)$ , which is 7-connected). Consider the  $n$ -connected

cover of its classifying space,  $(BX)\langle n \rangle$ . Theorem 5.1 implies then that  $QH^*((BX)\langle n \rangle; \mathbb{F}_p)$  is finitely generated as a module over  $\mathcal{A}_p$ . It belongs in fact to  $\mathcal{U}_{n-2}$ , compare with Corollary 4.6. For example  $QH^*(BDI(4)\langle 8 \rangle; \mathbb{F}_2)$  belongs to  $\mathcal{U}_6$ .

## APPENDIX A. MODULES AND ALGEBRAS

Let us consider an unstable algebra  $B^*$ . Assume there is another unstable algebra  $G^*$  which is finitely generated as an algebra over  $\mathcal{A}_p$  and which acts on  $B^*$ . Assume also that  $B^*$  is finitely generated as a module over  $G^*$ , when can we conclude that  $B^*$  is finitely generated as an algebra over  $\mathcal{A}_p$ ? If the action of  $G^*$  on  $B^*$  is compatible with the action of the Steenrod algebra, it is obvious, but it is not true in general as illustrated by the following example. Set  $G^* = H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$  and let  $B^*$  be isomorphic to  $G^*$  as an algebra, but define the action of  $\mathcal{A}_2$  to be trivial. Then  $B^*$  is finitely generated as a  $G^*$ -module, but not as an  $\mathcal{A}_2$ -algebra. We propose a weak notion of compatibility between the  $G^*$ -module and  $\mathcal{A}_p$ -algebra structures.

**Proposition A.1.** *Let  $G^*$  be a connected and commutative finitely generated  $\mathcal{A}_p$ -algebra, and let  $B^*$  be a connected commutative  $G^*$ -algebra which is a finitely generated as a  $G^*$ -module. Assume there exists a morphism  $p: B^* \rightarrow G^*$  of  $G^*$ -modules such that  $p(\theta(g \cdot 1)) = \theta g$  for all  $\theta \in \mathcal{A}_p$  and  $g \in G^*$ . Then  $B^*$  is also a finitely generated  $\mathcal{A}_p$ -algebra.*

*Proof.* Since  $B^*$  is finitely generated as a  $G^*$ -module, let us choose a finite set of generators  $b_1 = 1, \dots, b_n$ , where the degree of  $b_i$  is positive for  $i \geq 2$  by the connectivity assumption. In fact we will assume that  $b_i$  belongs to the kernel of  $p$  for  $i \geq 2$ : replace  $b_i$  by  $b_i - p(b_i) \cdot 1$  if necessary.

Let  $x$  be an element in  $B^*$ , so  $x$  can be written as a  $G^*$ -linear combination  $x = \sum_{i=1}^n \lambda_i \cdot b_i$ . If  $\{z_1, \dots, z_m\}$  denote the  $\mathcal{A}_p$ -algebra generators of  $G^*$ , then each  $\lambda_i$  can be expressed as a polynomial on Steenrod operations  $\theta_i$  applied to the generators,  $\theta_i z_i$  (use the Cartan formula). Our claim is that  $b_1, \dots, b_n$  and  $z_1 \cdot 1, \dots, z_m \cdot 1$  generate  $B^*$  as an algebra over  $\mathcal{A}_p$ . Since  $B^*$  is a unital  $G^*$ -algebra, we need to prove that an element of the form  $\theta z \cdot 1$ , for  $\theta \in \mathcal{A}_p$  and  $z \in \{z_1, \dots, z_m\}$ , can be written as a polynomial in the  $\theta(z_i \cdot 1)$ 's and the  $b_k$ 's.

If the action of the Steenrod algebra were compatible with the module action, one would have that  $\theta z \cdot 1 = \theta(z \cdot 1)$ . This is not the case, but it is sufficient to deal with the element  $\xi = \theta z \cdot 1 - \theta(z \cdot 1)$ . Note that  $\xi \in \text{Ker } p$  since  $p(\theta z \cdot 1) = \theta z$ . We will proceed by induction on the degree. If  $\xi$  is in degree zero, then the statement is clear since the algebras are connected. Assume that the statement is true for degrees  $< |\xi|$  and write  $\xi = \lambda_1 \cdot 1 + \lambda_2 \cdot b_2 + \dots + \lambda_n \cdot b_n$ . By the induction hypothesis we know that the elements  $\lambda_i \cdot 1$  can be expressed as polynomials in the  $\theta(z_i \cdot 1)$ 's and the  $b_k$ 's. So we only need to deal with  $\lambda_1 \cdot 1$ . But  $\lambda_1 = p(\lambda_1 \cdot 1) = p(\xi - \lambda_2 \cdot b_2 - \dots - \lambda_n \cdot b_n)$  which is zero because  $p$  is a morphism of  $G^*$ -modules and  $b_i \in \text{Ker } p$  for  $i \geq 2$ . This concludes the proof. ■

In Section 4 we need a slight generalization of this proposition.

**Proposition A.2.** *Let  $G^*$  be a connected and commutative finitely generated  $\mathcal{A}_p$ -algebra,  $C^*$  be a connected and finitely generated algebra, and let  $B^*$  be a connected commutative  $G^* \otimes C^*$ -algebra which is a finitely generated as a  $G^* \otimes C^*$ -module. Assume there exists a morphism  $p: B^* \rightarrow G^*$  of  $G^*$ -modules such that  $p(\theta(g \cdot 1)) = \theta g$  for all  $\theta \in \mathcal{A}_p$  and  $g \in G^*$ . Then  $B^*$  is also a finitely generated  $\mathcal{A}_p$ -algebra.*

*Proof.* Just as in the previous proof, the only problem is to write an element of the form  $(\theta z) \cdot 1$ , with  $\theta \in \mathcal{A}_p$  and  $z \in G^*$ , in terms of the generators  $z_i$ ,  $b_k$ , and generators  $c_m$  of the algebra  $C^*$ . The proof is then basically the same. ■

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